Electric Flux and Gauss Law

Objectives

After going through this lesson, the learners will be able to:

- Visualize the concept of Electric Flux.
- Understand the physical significance of Gauss Law
- Apply Gauss Law to find the electric field due to charge distribution
- Use the above knowledge to solve numerical on electric flux

Content Outline

- Unit Syllabus
- Module wise distribution of unit syllabus
- Words you must know
- Introduction
- Electric Flux density
- Gauss Law
- Application of Gauss Law to find electric field
- Summary

Unit Syllabus

Chapter-1: Electric Charges and Fields

Electric Charges; Conservation of charge, Coulomb's law- force between two point charges, forces between multiple charges; superposition principle and continuous charge distribution.

Electric field; electric field due to a point charge, electric field lines, electric dipole, electric field due to a dipole, torque on a dipole in uniform electric field.

Electric flux, statement of Gauss's theorem and its applications to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell (field inside and outside).

Chapter-2: Electrostatic Potential and Capacitance

Electric potential, potential difference, electric potential due to a point charge, a dipole and system of charges; equipotential surfaces, electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field.

Conductors and insulators, free charges and bound charges inside a conductor. Dielectrics and electric polarization, capacitors and capacitance, combination of capacitors in series and in parallel, capacitance of a parallel plate capacitor with and without dielectric medium between the plates, energy stored in a capacitor.

Module Wise Distribution Of Unit Syllabus

The above unit is divided into 11 modules for better understanding.

Module 1	• Electric charge • Properties of charge • Coulombs' law • Characteristics of coulomb force • Constant of proportionality and the intervening medium • Examples
Module 2	• Forces between multiple charges • Principle of superposition • Continuous distribution of charges • numerical
Module 3	• Electric field E • Importance of field E and ways of describing field • Superposition of electric field • Examples
Module 4	• Electric dipole • Electric field of a dipole • Charges in external field • Dipole in external field Uniform and non-uniform
Module 5	• Electric flux , • Flux density • Gauss theorem • Application of gauss theorem to find electric field • For a distribution of charges • Numerical
Module 6	• Application of gauss theorem Field due to field infinitely long straight wire

	• Uniformly charged infinite plane • Uniformly charged thin spherical shell (field inside and outside) • Graphs
Module 7	• Electric potential, • Potential difference, • Electric potential due to a point charge, a dipole and system of charges; • Equipotential surfaces, • Electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field. • Numerical
Module 8	• Conductors and insulators, • Free charges and bound charges inside a conductor. • Dielectrics and electric polarization
Module 9	• Capacitors and Capacitance, • Combination of capacitors in series and in parallel • Redistribution of charges , common potential • numerical
Module 10	• Capacitance of a parallel plate capacitor with and without dielectric medium between the plates • Energy stored in a capacitor
Module 11	• Typical problems on capacitors

Words You Must Know

Let us recollect the words we have been using in our study of this physics course.

- **Electric Charge:** Electric charge is an intrinsic characteristic of many of the fundamental particles of matter that gives rise to all electric and magnetic forces and interactions.
- **Conductors:** Some substances readily allow passage of electricity through them, others do not. Those which allow electricity to pass through them easily are called

conductors. They have electric charges (electrons) that are comparatively free to move inside the material. Metals, human and animal bodies and earth are all conductors of electricity.

- **Insulators:** Most of the non-metals, like glass, porcelain, plastic, nylon, wood, offer high opposition to the passage of electricity through them. They are called *insulators.*
- **Point Charge:** When the linear size of charged bodies is much smaller than the distance separating them, the size may be ignored and the charge bodies can then be treated as *point charges*.
- **Conduction:** Transfer of electrons from one body to another , it also refers to flow of charges electrons in metals and ions in electrolytes and gases
- **Induction:** The temporary separation of charges in a body due to a charged body in the vicinity. the effect lasts as long as the charged body is held close to the body in which induction is taking place
- **Quantization of charges:** Charge exists as an integral multiple of basic electronic charge. Charge on an electron is 1.6×10^{-19} C
- **Electroscope:** A device to detect charge
- Coulomb: S.I unit of charge defined in terms of 1 ampere current flowing in a wire to be due to 1 coulomb of charge flowing in 1 s
  1 coulomb = collective charge of 6×10^{18} electrons
- **Conservation of charge:** Charge can neither be created or destroyed in an isolated system it(electrons) only transfers from one body to another
- **Coulomb's Force:** It is the electrostatic force of interaction between the two point charges.
- **Vector form of coulomb's law:** A mathematical expression based on coulomb's law to show the magnitude as well as direction of mutual electrostatic force between two or more charges.
- **Laws of vector addition :** Triangle law of vector addition: If two vectors are represented by two sides of a triangle in order, then the third side represents the resultant of the two vectors
  Parallelogram law of vector addition: If two vectors are represented in magnitude and direction by adjacent sides of a parallelogram then the resultant of the vectors is given by the diagonal passing through their common point
  Also resultant of vectors P and Q acting at angle of θ is given by

$$R = \sqrt{P^2 + Q^2 + 2PQcos\theta}$$

Polygon law of vector addition: Multiples vectors may be added by placing them in order of a multi sided polygon, the resultant is given by the closing side taken in opposite order.

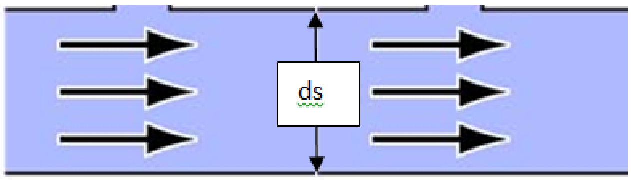
Resolution of vectors into components and then adding along x, y and z directions

- **Linear charge density:** The *linear charge density,* λ is defined as the charge per unit length.
- **Surface charge density:** The surface *charge density* σ is defined as the charge per unit surface area.
- **Volume charge density:** The volume ch*arge density* ρ is defined as the charge per unit volume.
- **Superposition Principle:** For an assembly of charges q_1, q_2, q_3, ..., the force on any charge, say q_1, is the vector sum of the force on q_1 due to q_2, the force on q_1 due to q_3, and so on. For each pair, the force is given by Coulomb's law for two point charges.
- **Torque:** Torque is the tendency of a force to rotate an object about an axis.
- Electric field lines: An electric field line is a curve drawn in such a way that the tangent at each point on the curve gives the direction of the electric field at that point.
- **Electric dipole:** Two equal but opposite charges separated by a small distance

  Dipole moment$p = 2aq$, $2a$ is the length of the dipole, p is in the direction from negative charge to positive
- **Dipole field:** Net charge on a dipole is zero but because of the small separation it has a field. Dipole field intensity is proportional to $1/r^3$.Dipoles experience a torque when placed in a uniform magnetic field.

Introduction

We often use the word 'flux' to talk about steady fluid flow through a uniform pipe:

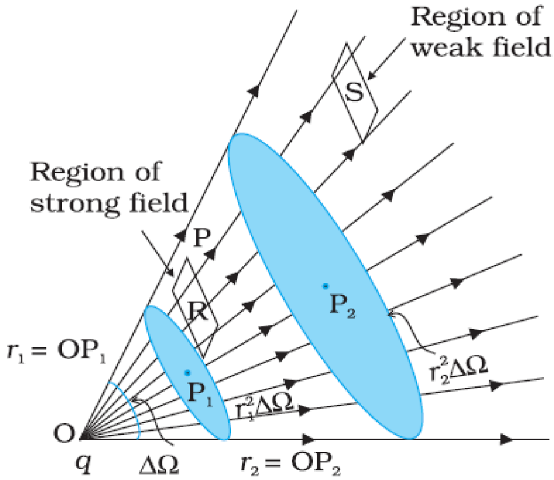


Consider flow of a liquid through a pipe of area of cross section dS with velocity v.

Now through a small flat surface dS, in a direction normal to the pipe the rate of flow of liquid is given by the **volume crossing the area per unit time** $v\,dS$ and represents the **flux** of liquid flowing across the plane.

If the normal to the surface is not parallel to the direction of flow of liquid, *i.e.*, v, but makes an angle θ with it, the projected area in a plane perpendicular to v is $v\,dS\,cos\,\theta$.

Therefore the flux going out of the surface dS is $v.\,n\,dS$ where n is the unit vector perpendicular to ds.



For the case of the electric field, we define an analogous quantity and call it **electric flux.**

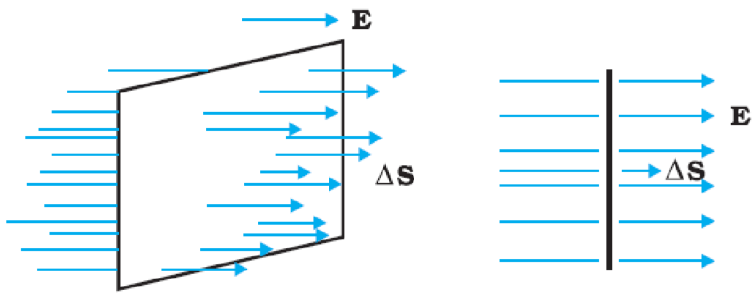
We should however note that there is no *flow* of a physically observable quantity unlike the case of liquid flow.

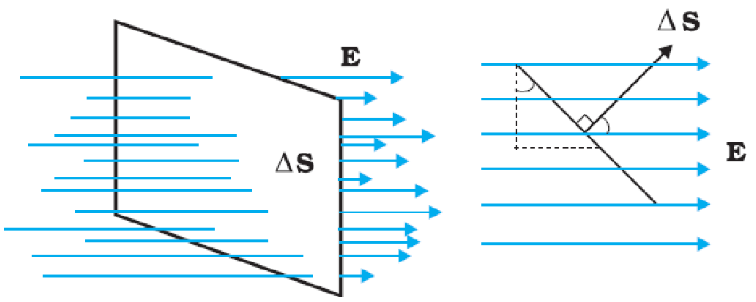
From our ideas about pictures of electric field lines, we saw that the number of field lines crossing a unit area, placed normal to the field at a point, is a measure of the strength of the electric field at that point.

This means that if we place a small planar element of area ΔS normal to E, at a point, the number of field lines crossing it is proportional to $E\,\Delta S$.

Now suppose we tilt the area element by angle θ clearly, the number of field lines crossing the area element will be smaller. The projection of the area element normal to E is $\Delta S\,cos\theta$.

Thus, the number of field lines crossing ΔS is proportional to $E\,\Delta S\,cos\theta$



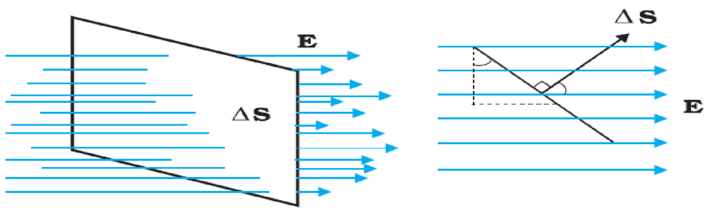


Or we can say θ is the angle between the field E and normal to the plane of the area.

When $\theta = 0°$, field lines will be perpendicular to ΔS

When $\theta = 90°$, field lines will be parallel to ΔS and will not cross it

Electric Flux Density



Dependence of flux on the inclination *ΔS* between E and nˆ

The orientation of an area element and not merely its magnitude is important in many contexts. For example, in a stream, the amount of water flowing through a ring will naturally depend on how you hold the ring. If you hold it normal to the flow, maximum water will flow through it than if you hold it with some other orientation.

This shows that an area element should be treated as a vector. It has a magnitude and also a direction.

How to specify the direction of a planar area?

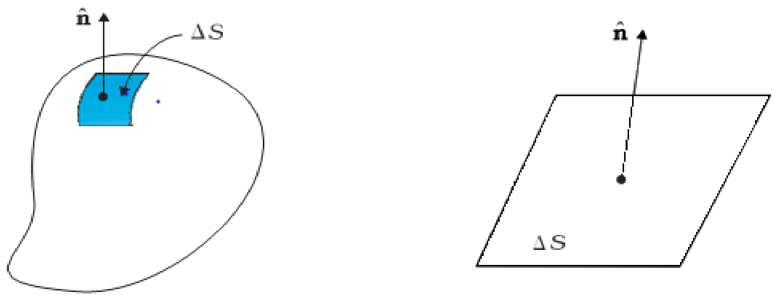
Clearly, the normal to the plane specifies the orientation of the plane. Thus the direction of a planar area vector is along its normal.

How to associate a vector to the area of a curved surface? We imagine dividing the surface into a large number of very small area elements.

Each small area element may be treated as planar and a vector associated with it, as explained before.

The direction of an area element is along its normal. But a normal can point in two directions. Which direction do we choose as the direction of the vector associated with the area element? This problem is resolved by some convention appropriate to the given context. For the case of a closed surface, this convention is very simple.

The vector associated with every area element of a closed surface is taken to be in the direction of the *outward* normal.



Electric flux due to an enclosed area

Thus, the area element vector ΔS at a point on a closed surface equals ΔS ^ where ΔS is the magnitude of the area element and ^ is a unit vector in the direction of outward normal at that point.

We now come to the definition of electric flux.

Electric flux Δϕφ through an area element ΔS is defined by:

$\Delta\phi = E.\ \Delta S = E\Delta S\ cos\theta$

which, as seen before, is proportional to the number of field lines cutting the area element. The angle θ here is the angle between E and ΔS. For a closed surface, with the convention stated already, θ is the angle between E and the outward normal to the area element.

See the animated video with the module to understand electrostatic flux, area vector.

Notice

we could look at the expression $E\,\Delta S\,cos\theta$ in two ways: $E\,(\Delta S\,cos\theta\,)$ i.e., E times the projection of area normal to E, or $E\perp\,\Delta S$, i.e., the component of E along the normal to the area element times the magnitude of the area element.

The unit of electric flux is $NC^{-1}m^{2}$**.**

The basic definition of electric flux given by the equation can be used, in principle, to calculate the total flux through any given surface. All we have to do is to divide the surface into small area elements, calculate the flux at each element and add them up.

Thus, the total flux ϕthrough a surface S is:

$$\phi\cong\sum E.\ \Delta S$$

The approximation sign is put because the electric field E is taken to be constant over the small area element. This is mathematically exact only when you take the limit $\Delta S\rightarrow 0$ and the sum in is written as an integral.

$$\phi\ =\int E.ds$$

Example:

An electric field E=3$\hat{}$ + 4$\hat{}$ + 3$\hat{}$ NC^{-1} is applied to an area of 100 units in the x- y plane. Find the electric flux linking with this area?

Solution:

As normal to area will lie along z axis this implies-

$\Delta S\ =\ 100k$

So electric flux linking with the surface $\Delta\phi\ \ =\ \ E.\Delta S$ **is-**

$\Delta\phi\ =\ (3\hat{}\ +\ 4\hat{}\ +\ 3\hat{}).100k\ =\ 300\,units$

Example:

Consider a uniform electric field $E\ \ =\ \ 3\times10^{3}\,\hat{\imath}\,N/C$.

- What is the flux of this field through a square of 10 cm on a side whose plane is parallel to the y-z?
- What is the flux through the same square if the normal to its plane makes a 60° angle with the *x*-axis?

Solution:

- $\Delta\phi = E.\Delta S = E\,\Delta S\cos\theta$

$$= 3\times10^{3}\times10^{-2}\cos 0 \quad = 30\ Nm/C^{2}$$

- $\Delta\phi = E.\Delta S = E\,\Delta S\cos\theta$

$$= 3\times10^{3}\times10^{-2}\cos 60 \quad = 15\ \text{Nm/C}^{2}$$

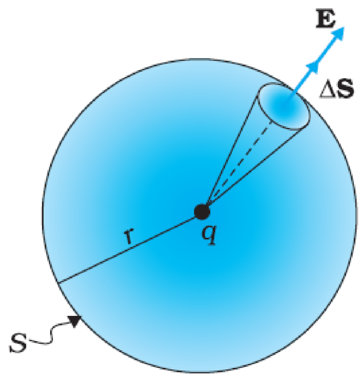
If there is a charged body or a charge distribution, there is a region around it where its electrical influence is experienced by test charges or charged bodies. The region is called the electric field. The intensity of the field is given by force per unit charge at that point in the field of by field lines crossing a certain area. The collection of field lines per unit perpendicular to the field lines gives the electric flux.

flux is given by, $\Delta\phi = E.\,\Delta S = E\Delta S\cos\theta$

Both the ideas of electric field and flux are important to us.

Gauss's Law

As a simple application of the notion of electric flux, let us consider the total flux through a sphere of radius *r*, which encloses a point charge *q* at its center. Divide the sphere into small area elements, as shown in Fig.



The flux through area element ΔS is-

$$\Delta\phi = E.\Delta S = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.\Delta s$$

Here, we have used Coulomb's law for the electric field due to a single charge *q*.

The unit vector $\hat{r}$ is along the radius vector from the center to the area element.

Now, since the normal to a sphere at every point is along the radius vector at that point, the area element ΔS and $\hat{r}$ have the same direction. Therefore,

$$\Delta\phi = \frac{q}{4\pi\varepsilon_0 r^2}\Delta s$$

Since the magnitude of a unit vector is 1.

The total flux through the sphere is obtained by adding up flux through all the different area elements:

$$\phi = \sum_{all\ \Delta S} \frac{q}{4\pi\varepsilon_0 r^2} \Delta s$$

Since each area element of the sphere is at the same distance *r* from the charge:

$$\phi = \frac{q}{4\pi\varepsilon_0 r^2} \sum_{all\ \Delta S} \Delta s = \frac{q}{4\pi\varepsilon_0 r^2} s$$

Now *S*, the total area of the sphere, equals $4\pi r^2$. Thus:

$$\phi = \frac{q}{4\pi\varepsilon_0 r^2} \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

This is a simple illustration of a general result of electrostatics called **Gauss's law**.

We state *Gauss's law* without proof:

Electric flux through a closed surface S:

$$\phi = q/\varepsilon_0$$

Where q = total /net charge enclosed by S.

The permittivity of free space ε_0 suggests that the charge q is placed in vacuum

The law implies that the total electric flux through a closed surface is zero if no charge is enclosed by the surface.

We can see that explicitly in the simple situation:

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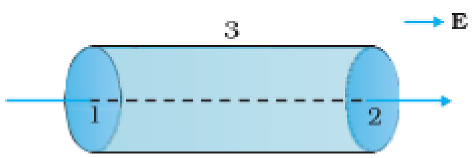


Here the electric field is uniform and we are considering a closed cylindrical surface, with its axis parallel to the uniform field E.

The total flux φ through the surface is $\phi = \phi_1 + \phi_2 + \phi_3$, where ϕ_1 and ϕ_2 represent the flux through the surfaces 1 and 2 (of circular cross-section) of the cylinder and ϕ_3 is the flux through the curved cylindrical part of the closed surface.

Now the normal to the surface 3 at every point is perpendicular to E, so by definition of flux,

$$_3 = 0.$$

Further, the outward normal to 2 is along E while the outward normal to 1 is opposite to E. Therefore,

$$_1 = -E\,S_1, \quad _2 = +\,E\,S_2$$

$S_1 = S_2 = S$ where S is the area of circular cross-section. Thus, the total flux is zero, as expected by Gauss's law.

Thus, **whenever you find that the net electric flux through a closed surface is zero, we conclude that the total charge contained in the closed surface is zero.**

Gauss's law is applicable only under following two conditions:

- The electric field at every point on the surface is either perpendicular or tangential.
- Magnitude of the electric field at every point where it is perpendicular to the surface has a constant value (say *E*).

The great significance of Gauss's law is that it is true in general, and not restricted only to the simple cases we have considered above.

Let us note some important points regarding this law:

- Gauss's law is true for any closed surface , no matter what its shape or size
- In the expression $\phi = q/\varepsilon_0$, q includes the sum of all charges enclosed by the surface. The charges may be located anywhere in the surface
- In the situation when the surface is so chosen that there are some charges inside and some outside, the electric field [whose flux appears on the left side of gauss's law is due to all the charges, both inside and outside *s*. The term *q* on the right side of gauss's law, however, represents only the total charge inside *s*.
- Gauss's law is often useful towards a much easier calculation of the electrostatic field *when the system has some symmetry.*

  The electric field due to a general charge distribution can be calculated as we have seen .This method uses summation or integration, which cannot be carried out to give electric fields at every point in space. For some symmetric charge distributions gauss's law can be used easily.
- This is facilitated by the choice of a suitable Gaussian surface, the surface that we choose for application of Gauss's law.

- The surface should be chosen in a way that the Gaussian surface should not pass through any discrete charge.
- Gauss's law is based on the inverse square dependence on distance contained in Coulomb's law. Any violation of gauss's law will indicate departure from the inverse square law. Gauss's law is true for any closed surface, no matter what its shape or size.
- The surface that we choose for the application of gauss's law is called the Gaussian surface. You may choose any Gaussian surface and apply Gauss's law. However, take care not to let the Gaussian surface pass through any discrete charge. This is because the electric field due to a system of discrete charges is not well defined at the location of any charge. (As you go close to the charge, the field grows without any bound
- The Gaussian surface can pass through a continuous charge distribution.
- The electric flux through a closed surface can be zero, positive and negative.
- If the total charge is positive then electric flux is positive. This means electric field lines are leaving the surface.
- If the total charge is negative then electric flux is negative. This means electric field lines are entering the surface.
- The zero electric flux means the number of electric field lines entering the surface is equal to the number of field lines leaving the surface.

Example:

If Coulomb's law involved $1/r^3$ dependence instead of $1/r^2$ Would Gauss' law still be true?

Solution:

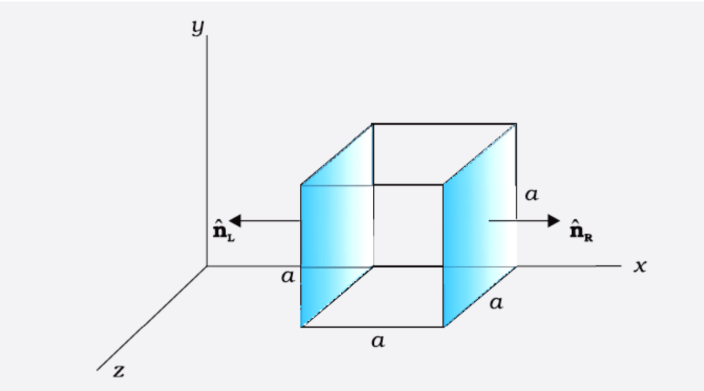
Gauss won't be valid. Because then the R.H.S of Gauss law would not be independent of size and shape of Gaussian surface.

Example:

The electric field components in Fig. are $E_x = \alpha x^{1/2}$, $E_y = E_z = 0$, in which α = 800 N/C $m^{1/2}$.

Calculate

- The flux through the cube, and



- The charge within the cube. Assume that a = 0.1 m.

Solution:

Since the electric field has only an x component, for faces perpendicular to x direction, the angle between E and ΔS is ± 90^0

Therefore, the flux $= E.\Delta S$ is separately zero for each face of the cube except the two shaded ones.

Now the magnitude of the electric field at the left face is:

$$E_L = \alpha x^{1/2} = \alpha a^{1/2}$$

($x = a$, at the left face). The magnitude of electric field at the right face is:

$$E_R = \alpha x^{1/2} = \alpha(2a)^{1/2}$$

($x = 2a$ At the right face). The corresponding fluxes are:

$$\phi_L = E_L.\Delta S = \Delta S E_L.\hat{n}_L = E_L \Delta S \cos\theta = -E_L \Delta S \quad (since\ \theta = 180^0)$$

$$\phi_L = -E_L a^2$$

$$\phi_R = E_R.\Delta S = E_R \Delta S \cos\theta = E_R \Delta S \quad (since\ \theta = 0^0)$$

$$\phi_R = E_R a^2$$

Net flux through the cube:

$$= \phi_L + \phi_R = E_R a^2 - E_L a^2 = a^2\left(E_R - E_L\right)$$

$$= a^2\left(\alpha(2a)^{\frac{1}{2}} - \alpha a^{\frac{1}{2}}\right) = \alpha a^{5/2}\left(\sqrt{2} - 1\right) = 800\ (0.1)^{5/2}\left(\sqrt{2} - 1\right) = 1.05\ Nm^2C^{-1}$$

(b) We can use Gauss's law to find the total charge q inside the cube. We have:

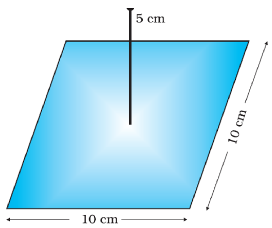
$$\phi = q/(\varepsilon_0)\ or\quad q = \phi\varepsilon_0$$

Therefore,

q = 1.05 × 8.854 ×10^{-12} C = 9.27 ×10^{-12} C

Problems For Practice

- Consider a uniform electric field $E = 2 \times 10^3 \hat{\imath}\ N/C$.
  - What is the flux of this field through a square of 20 cm on a side whose plane is parallel to the $y - z$ plane?
  - What is the flux through the same square if the normal to its plane makes a 90° angle with the x-axis?
- What is the net flux of the uniform electric field of problem 1 through a cube of side 20 cm oriented so that its faces are parallel to the coordinate planes?
- Careful measurement of the electric field at the surface of a black box indicates that the net outward flux through the surface of the box is $8.0 \times 10^3\ Nm^2/C$.
  - What is the net charge inside the box?
  - If the net outward flux through the surface of the box were zero, could you conclude that there were no charges inside the box? Why or Why not?
- A point charge $+ 10\ \mu C$ is a distance 5 cm directly above the centre of a square of side 10 cm, as shown in Fig



What is the magnitude of the electric flux through the square?
(*Hint:* Think of the square as one face of a cube with edge 10 cm)

Application of Gauss Law to Find an Electric Field

In principle Gauss's law is valid for the electric field of any system of charges or continuous distribution of charge. In practice however, the technique is useful for calculating the electric field only in situations where the degree of symmetry is high.

Gauss's law can be used to evaluate the electric field for charge distributions that have spherical, cylindrical or plane symmetry such that over a closed surface (Gaussian surface) chosen, the magnitude of the field remains constant and E can be taken out from summation or integration.

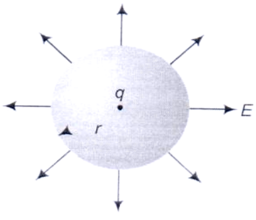
As an example we take a simple case

Electric Field Due to a Point Charge

As we know that the electric field due to a point charge is everywhere radial as shown in figure.

We wish to find the electric field at a distance r from the charge q. We select a Gaussian surface, a sphere at distance r from the charge. At every point of this sphere the electric field has the same magnitude E and it is perpendicular to the surface itself. Hence, we can apply the simplified form of Gauss law,

$$E.\Delta S \ = \ _1/(\varepsilon_0)$$



A point charge with electric field lines

i.e.

$$E4\pi r^2 \ = \ _1/\varepsilon_0$$

$$E = \frac{1}{4\pi\varepsilon_0 v^2}$$

This gives the magnitude of the electric field at a distance r from the point charge. If we put a test charge $_1$ at that point then the magnitude of force experienced by test charge will be:

$$F = Eq = \frac{_1q}{4\pi\varepsilon_0 v^2}$$

This is Coulomb's law derived from Gauss law. Thus Coulomb's law and Gauss law in electrostatics are mutually equivalent.

Summary

- The flux $\Delta\phi$ of electric field E through small area element ΔS is given by

  $$\Delta\phi = E.\Delta S$$

  The vector area element ΔS is $\Delta S = \Delta S\ \hat{n}$ where ΔS is the magnitude of the area element and $\hat{n}$ is normal to the area element, which can be considered planar for sufficiently small ΔS. For an area element of a closed surface, $\hat{n}$ is taken to be the direction of *outward* normal, by convention.

- *Gauss's law*: The flux of electric field through any closed surface S is $1/\varepsilon_0$ times the total charge enclosed by S. The law is especially useful in determining electric field E, when the source distribution has simple symmetry like cylindrical or spherical.
- Gauss's law is true for any closed surface, no matter what its shape or size. The term q on the right side of Gauss's law includes the sum of all charges enclosed by the surface. The charges may be located anywhere inside the surface.